SOLUTIONS OF INTEGRAL EQUATIONS OF TRANSPORT THEORY IN THE GREEN-FUNCTION FORMALISM

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The uniqueness of solutions of some integral equations of transport theory is analyzed for a specific shape of the kernels of these equations.

§I. Methods of determining the scattering amplitudes of several quantum-mechanical particles were developed in great detail in recent years. These methods are based on Fredholm integral equations of the second kind, described in the Green-function formalism. The systems of integral equations obtained in this formalism are equivalent to the Lippman–Schwinger equation [1], formally generalized to an N-body system under the assumption of two-particle forces. The uniqueness and equivalence, however, of the solutions of these systems of equations to the solution of the corresponding Schrödinger equation was not proved.

In the present paper examples are given for which in the case of scattering of four or more particles the homogeneous integral equation for the scattering amplitude, obtained by the Green-function method, has an infinite number of solutions, and, consequently, the inhomogeneous equation also has an infinite number of solutions.

Consider first the Lippman–Schwinger equation for the four-body problem, when two independent pairs of particles (12) and (34) interact through the potentials V_{12} and V_{34}, respectively [2]. This equation is [1]

$$T(z) = (V_{12} + V_{34}) + (V_{12} + V_{34}) G_0(z) T(z), \tag{1}$$

where T(z) is the particle scattering operator in the system, $G_0(z)$ is the Green function of free motion of four particles, and z is the total energy of the system.

It is well known that for this problem the Schrödinger equation has a unique solution, written as follows in terms of the scattering operators:

$$T(z') = t_{12}(z_{12}) + t_{34}(z_{34}) + t_{12}(z_{12}) \otimes t_{34}(z_{34}), \tag{2}$$

where $t_{12}(z_{12})$ and $t_{34}(z_{34})$ are the scattering operators of particle pairs (12) and (34), respectively; z_{12} and z_{34} are the particle energies of relative motion in the systems of particles (12) and (34); $z' = z - z_{12,34}$, $z' = z_{12} + z_{34}$, $z_{12,34}$ are the energies of relative motion of the two systems of particles (12) and (34); and the symbol $\otimes$ denotes a product of scattering operators. We point out that due to the absence of particle exchange between the systems (12) and (34), each of the energies z_{12} and z_{34} is conserved in the scattering process.

To write down the integral equation for the amplitude of particle scattering in the system it is necessary to determine the vectors of initial and final states in the four-particle system. For this purpose we use the Jacobi coordinates in the momentum representation. The vectors of initial and final states of the four free particles can then be written in the form

$$\langle \mathbf{k}_{12}, \mathbf{k}_{34}, \mathbf{k}_{12,34}|, \quad |\mathbf{k}'_{12}, \mathbf{k}'_{34}, \mathbf{k}'_{12,34}\rangle,$$

respectively. Here $\mathbf{k}_\alpha$ and $\mathbf{k}'_\alpha$ are the momenta of relative motion of particles in the system (α) prior to and following scattering, and $\mathbf{k}_{12,34}$ and $\mathbf{k}'_{12,34}$ are the momenta of relative motion of the systems (12) and (34) prior to and following scattering. In this case

$$z_{12} = k_{12}^2/2\mu_{12};\ z_{34} = k_{34}^2/2\mu_{34},\ z_{12,34} = k_{12,34}^2/2\mu_{12,34},$$

where μ_α is the reduced mass of particles in system α, and $\mu_{12,34}$ is the reduced mass of four particles.

In these coordinates the matrix element of the operator T(z) in (1) is diagonal in the variable $\mathbf{k}_{12,34}$:

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$$<\mathbf{k}_{12},\ \mathbf{k}_{34},\ \mathbf{k}_{12,34}|T(z)|\mathbf{k}'_{12},\ \mathbf{k}'_{34},\ \mathbf{k}'_{12,34}> = <\mathbf{k}_{12},\ \mathbf{k}_{34}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}>\delta(\mathbf{k}_{12,34}-\mathbf{k}'_{12,34}).$$

We use a model [3] in which an integral equation is written down for the matrix element of the many-particle scattering operator. In this model the matrix element of the kernel of Eq. (1) is

$$<\mathbf{k}_{12},\ \mathbf{k}_{34}|V_{12}G_0(z')|\mathbf{q}_{12},\ \mathbf{q}_{34}> = <\mathbf{k}_{12}|V_{12}|\mathbf{q}_{12}>\ (z_{12}-q_{12}^2/2\mu_{12}+i\tau)^{-1}\delta(\mathbf{k}_{34}-\mathbf{q}_{34}). \tag{3}$$

One can then write, on the basis of (1), an integral equation for the amplitude $<\mathbf{k}_{12},\ \mathbf{k}_{34}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}>$ at the energy surface in the form

$$<\mathbf{k}_{12},\ \mathbf{k}_{34}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}> = (2\pi)^3<\mathbf{k}_{12}|V_{12}|\mathbf{k}'_{12}>\delta(\mathbf{k}_{34}-\mathbf{k}'_{34}) + (2\pi)^3<\mathbf{k}_{34}|V_{34}|\mathbf{k}'_{34}>\delta(\mathbf{k}_{12}-\mathbf{k}'_{12}) +$$
$$+\int\frac{d\mathbf{q}_{12}}{(2\pi)^3}\ \frac{<\mathbf{k}_{12}|V_{12}|\mathbf{q}_{12}>}{z'-k_{34}^2/2\mu_{34}-q_{12}^2/2\mu_{12}+i0}<\mathbf{k}_{34},\ \mathbf{q}_{12}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}>+$$
$$+\int\frac{d\mathbf{q}_{34}}{(2\pi)^3}\ \frac{<\mathbf{k}_{34}|V_{34}|\mathbf{q}_{34}>}{z'-k_{12}^2/2\mu_{12}-q_{34}^2/2\mu_{34}+i0}<\mathbf{k}_{12},\ \mathbf{q}_{34}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}>. \tag{4}$$

We note that the denominators in the integral terms of Eq. (4) can be written in the equivalent form

$$z'-k_{34}^2/2\mu_{34}-q_{12}^2/2\mu_{12} = k_{12}^2/2\mu_{12}-q_{12}^2/2\mu_{12}+i0,$$
$$z'-k_{12}^2/2\mu_{12}-q_{34}^2/2\mu_{34} = k_{34}^2/2\mu_{34}-q_{34}^2/2\mu_{34}+i0.$$

We investigate Eq. (4) for the case in which the potentials are separable [2] and the particle masses are equal:

$$<\mathbf{k}_{12}|V_{12}|\mathbf{k}'_{12}> = \frac{-G}{2\mu}g(k_{12})g(k'_{12}),$$
$$<\mathbf{k}_{34}|V_{34}|\mathbf{k}'_{34}> = \frac{-\Omega}{2\mu}\omega(k_{34})\omega(k'_{34}) \tag{5}$$

[the constants G and Ω, as well as the functions g(k) and ω(k), are assumed real, which guarantees potentials being Hermitian]. The pair amplitudes are hence also separable:

$$<\mathbf{k}_{12}|t_{12}(k_{12}^2/2\mu)|\mathbf{k}'_{12}> = -\frac{G}{2\mu}g(k_{12})g(k'_{12})\left[1+\frac{G}{2\mu}\int\frac{d\mathbf{q}_{12}}{(2\pi)^3}\ \frac{g^2(q_{12})}{k_{12}^2/2\mu-q_{12}^2/2\mu+i0}\right.;$$
$$<\mathbf{k}_{34}|t_{34}(k_{34}^2/2\mu)|\mathbf{k}'_{34}> = -\frac{\Omega}{2\mu}\omega(k_{34})\omega(k'_{34})\left[1+\frac{\Omega}{2\mu}\int\frac{d\mathbf{q}_{34}}{(2\pi)^3}\ \frac{\omega^2(q_{34})}{k_{34}^2/2\mu-q_{34}^2/2\mu+i0}\right.. \tag{6}$$

Relationships (6) form the analytic solution [2] of the corresponding Lippman–Schwinger integral equations.

We introduce the following notation, which will be required in the sequel:

$$a(k_{12}^2/2\mu) = \frac{G}{2\mu}\int\frac{d\mathbf{q}_{12}}{(2\pi)^3}\ \frac{g^2(q_{12})}{k_{12}^2/2\mu-q_{12}^2/2\mu+i0},$$
$$b(k_{34}^2/2\mu) = \frac{\Omega}{2\mu}\int\frac{d\mathbf{q}_{34}}{(2\pi)^3}\ \frac{\omega^2(q_{34})}{k_{34}^2/2\mu-q_{34}^2/2\mu+i0}. \tag{7}$$

Obviously, the pair amplitudes (6) exist if the following conditions are satisfied:

$$1+a(k_{12}^2/2\mu)=0,\quad 1+b(k_{34}^2/2\mu)=0. \tag{8}$$

We write down the homogeneous equation corresponding to expression (4):

$$<\mathbf{k}_{12},\ \mathbf{k}_{34}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}> = \int\frac{d\mathbf{q}_{12}}{(2\pi)^3}\ \frac{<\mathbf{k}_{12}|V_{12}|\mathbf{q}_{12}>}{k_{12}^2/2\mu-q_{12}^2/2\mu+i0}<\mathbf{q}_{12},\ \mathbf{k}_{34}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}>+$$
$$+\int\frac{d\mathbf{q}_{34}}{(2\pi)^3}\ \frac{<\mathbf{k}_{34}|V_{34}|\mathbf{q}_{34}>}{k_{34}^2/2\mu-q_{34}^2/2\mu+i0}<\mathbf{q}_{34},\ \mathbf{k}_{12}|T(z')|\mathbf{k}'_{12},\ \mathbf{k}'_{34}>. \tag{9}$$

Since the potentials are assigned by relations (5), and the energies z_{12} and z_{34} are conserved, the solution (9) is of the form

$$\langle \mathbf{k}_{12}, \mathbf{k}_{34} | T(z') | \mathbf{k}'_{12}, \mathbf{k}'_{34} \rangle = g(k_{12})\, g(k'_{12})\, \omega(k_{34})\, \omega(k'_{34})\, F(z'), \tag{10}$$

$$z' = z_{12} + z_{34}, \quad z_{12} = k_{12}^2/2\mu, \quad z_{34} = k_{34}^2/2\mu,$$

if conditions (8) are satisfied.

Substituting (10) into Eq. (9), the following equation is obtained for the function $F(z')$:

$$F(z')\,[1 + a(z_{12}) + b(z_{34})] = 0. \tag{11}$$

It is seen from (11) that the homogeneous equation (9) can have an infinite number of solutions [$F(z')$ is an arbitrary function] if the following condition is satisfied:

$$[1 + a(z_{12}) + b(z_{34})] = 0. \tag{12}$$

This implies that the inhomogeneous equation (4) also has no unique solution if condition (12) is satisfied.

A necessary and sufficient condition for the existence of a unique (trivial) solution of the homogeneous equation (9) follows from Eq. (11):

$$[1 + a(z_{12}) + b(z_{34})] \neq 0, \tag{13}$$

if $1 + a(z_{12}) \neq 0$; $1 + b(z_{34}) \neq 0$.

§II. We further show that a class of separable potentials can be found with parameters for which condition (12) is satisfied for the physical energy region $z' = E + i0$. For simplicity we assume that $k_{12} = k_{34} = \varkappa$, $z' = \varkappa^2/\mu + i0$.

Our problem is to find examples of potentials for which the condition $1 + a(\varkappa) + b(\varkappa) = 0$, where

$$a(\varkappa) = \frac{G}{2\mu} \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{g^2(\mathbf{q})}{\varkappa^2/2\mu - q^2/2\mu + i0},$$

$$b(\varkappa) = \frac{\Omega}{2\mu} \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{\omega^2(\mathbf{q})}{\varkappa^2/2\mu - q^2/2\mu + i0}. \tag{14}$$

According to the Sokhotskii equation, the following representation can be obtained for $a(\varkappa)$ and $b(\varkappa)$ from (14):

$$a(\varkappa) = \alpha(\varkappa) + i\beta(\varkappa); \quad b(\varkappa) = \delta(\varkappa) + i\varphi(\varkappa), \tag{15}$$

where

$$\alpha(\varkappa) = \frac{G}{(2\pi)^2} Vp \int_0^\infty dq \frac{q^2 g^2(q)}{\varkappa^2 - q^2}; \tag{16}$$

$$\beta(\varkappa) = -\frac{1}{4\pi} G\varkappa^2 g^2(\varkappa); \tag{17}$$

$$\delta(\varkappa) = \frac{\Omega}{2\pi^2} Vp \int_0^\infty dq \frac{q^2 \omega^2(q)}{\varkappa^2 - q^2}; \tag{18}$$

$$\varphi(\varkappa) = -\frac{1}{4\pi} \Omega\varkappa^2 \omega^2(\varkappa). \tag{19}$$

To satisfy the condition $1 + a(\varkappa) + b(\varkappa) = 0$, it is necessary to require that

$$-\beta(\varkappa) = \varphi(\varkappa), \quad 1 + \alpha(x) + \delta(\varkappa) = 0. \tag{20}$$

These conditions automatically guarantee the existence of nontrivial solutions of Eqs. (9) for real G and Ω, i.e., Hermitian potentials (7).

Let, for example, the functions $g(q)$ and $\omega(q)$ be

$$g(q) = \frac{\sqrt{q}}{\lambda^2 + q^2}, \quad \omega(q) = \frac{\sqrt{q}}{\eta^2 + q^2}. \tag{21}$$

Calculations of the integrals in (16)-(19) show that for certain relations between the parameters λ, η and the system energy $\varkappa^2/\mu$ conditions (20) are satisfied.

We have thus shown that for a given separable potential of the form (21) an energy value is found for which the homogeneous equation (9) has an infinite number of solutions. Hence, the corresponding inhomogeneous equation (4) also has an infinite number of solutions.

We point out that the satisfaction of condition (12) for separable potentials leads to the result that the system of integral equations derived from the operator equations [3],

$$T_{12}(z') = t_{12}(z_{12}) + t_{12}(z_{12}) G_0(z') T_{34}(z'),$$
$$T_{34}(z') = t_{34}(z_{34}) + t_{34}(z_{34}) G_0(z') T_{12}(z'),$$

where

$$T_{12}(z') + T_{34}(z') = T(z')$$

and T(z') satisfies Eq. (2), has the same properties as does the original equation (1). The corresponding homogeneous system can have an infinite number of solutions, and the inhomogeneous system can have an infinite number of solutions or be incompatible. The results of this study can be generalized to the scattering problem of five or more bodies under the assumption of pair interactions.

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GENERALIZATION OF SOMMERFELD HEAT-CONDUCTION PROBLEM FOR A RING

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A solution is obtained for the problem of heat conduction in a one-dimensional ring consisting of two sections with different lengths, heat sources, and thermophysical parameters.

The problem of the heat conduction in a ring [1] is an example of a boundary-value problem in which there are no boundary conditions of the first, second, and third kinds modeling the effect of the external medium on the system. In view of the symmetry of the problem, these conditions are replaced by the periodicity condition for the solution. Such "self-closed" systems may serve as mathematical models of different processes of heat and mass transfer [2, 3].

§1. Consider the problem of determining the temperature field in a one-dimensional composite ring, the n sections of which have different lengths, thermophysical parameters, and heat sources. Any of the sections may be regarded as a system interacting with its "environment" – the other sections. The initial temperature distribution in the different sections of the ring is described by different functions and is discontinuous at the contact points, where boundary conditions of the fourth kind are assumed. Since a one-dimensional problem is considered, the shape of the ring is unimportant, as in [1]. A linear coordinate x_i is introduced for each section, $x_i \in (0, l_i)$, $i = 1, 2, \ldots, n$. The mathematical formulation of the linear heat-conduction problem for a composite ring takes the form

$$\frac{\partial \tilde{T}_i}{\partial t} - a_i \frac{\partial^2 \tilde{T}_i}{\partial x_i^2} = \tilde{f}_i + \varphi_i \delta(t), \quad t \geqslant 0, \quad x_i \in (0, l_i), \tag{1.1}$$

$$\tilde{T}_i = \Theta(t) T_i(x_i, t), \quad \tilde{f}_i = \Theta(t) f_i(x_i, t), \quad a_i = \lambda_i/(\rho c)_i,$$

$$\varphi_i = \varphi_i(x_i), \quad \delta(t) = \frac{d\Theta}{dt}, \quad \Theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leqslant 0, \end{cases}$$

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